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Zero-Temperature Dynamics of Ising Models on the Triangular Lattice

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We consider the nature of spin flips of zero-temperature dynamics for ferromagnetic Ising models on the triangular lattice with nearest-neighbor interactions and an initial configuration chosen from a symmetric Bernoulli distribution. We prove that all spins flip infinitely many times for almost every realization of the dynamics and initial configuration.

KEY WORDS: Ising models; triangular lattice; spin flips.

1. INTRODUCTION

Let $\mathcal{G} = (E_{\mathcal{G}}, V_{\mathcal{G}})$ be an infinite graph with edge set $E_{\mathcal{G}}$ and vertex set $V_{\mathcal{G}}$. Consider the stochastic process $\sigma^t = \{\sigma_x^t, x \in V_{\mathcal{G}}\}$ on $\mathcal{G}$ which corresponds to the zero-temperature limit of Glauber dynamics for the Ising model with Hamiltonian

$$H = -\sum_{x, y : x \sim y} J_{x, y} \sigma_x \sigma_y,$$

where $x \sim y$ denotes that x and y are nearest neighbors, i.e., $\{x, y\} \in E_{\mathcal{G}}$. The process σ^t takes values in $\Omega = \{-1, +1\}^{V_{\mathcal{G}}}$, the space of spin configurations. The initial value $\sigma^0 = \{\sigma_x^0, x \in V_{\mathcal{G}}\}$ is taken from a symmetric Bernoulli product measure, i.e., for each $x \in V_{\mathcal{G}}$, σ_x^0 takes $+1$ (or -1) with probability $1/2$, independently for different vertex. At each x, there is a "Poisson clock" (with rate 1) which "rings" at random times independently

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between different vertices. When the Poisson clock rings at time t, the spin at x will consider to flip (i.e., $\sigma_x^{t+0} = -\sigma_x^{t-0}$). If the *change in energy*

$$\triangle H_x(\sigma) = 2 \sum_{y:y\sim x} J_{x,y}\sigma_x\sigma_y$$

(where $\sigma = \sigma^{t-0}$) is negative (or positive), then the spin at x flips (or does not flip) with probability 1. If $\triangle H_x(\sigma) = 0$, then a fair coin is tossed to determine if the spin at x flips. We denote by P_τ the probability distribution on the times at which the Poisson clock rings along with the fair coin tosses, and by $P_{\sigma^0\times\tau} = P_{\sigma^0}\times P_\tau$ the joint distribution of the independent σ^0 and τ.

We are interested in whether for almost every σ^0 and τ (i.e., $P_{\sigma^0\times\tau}$-almost surely), $\sigma^\infty(\sigma^0,\tau) = \lim_{t\to\infty}\sigma^t(\sigma^0,\tau)$ exists, that is, whether for every $x\in V_{\mathscr{G}}$, $\sigma_x^t(\sigma^0,\tau)$ flips only finitely many times. If $J_{x,y}$'s are independent random variables and $\mathscr{J} = \{J_{x,y}, \{x,y\}\in E_{\mathscr{G}}\}$ is chosen from the product measure $P_{\mathscr{J}}$ of a continuously distributed probability measure on the real line, then it is proved in ref. 2, under very mild conditions, that for almost every $\mathscr{J}$, σ^0, and τ, $\sigma^t(=\sigma^t(\sigma^0,\tau))$ flips only finitely many times for every $x\in V_{\mathscr{G}}$, for a very general class of infinite graphs, including the hypercubic lattice Z^d, the hexagonal and triangular lattice, homogeneous trees, etc. On the other hand, if $J_{x,y} = +1$ (or a positive constant) for all $\{x,y\}\in E_{\mathscr{G}}$, then the picture is less complete. In this case it is proved in ref. 2 that (1) σ_x^t flips infinitely many times for almost every σ^0 and τ and for every $x\in V_{\mathscr{G}}$ if $\mathscr{G} = Z$ or Z^2, and (2) σ_x^t flips only finitely many times for almost every σ^0 and τ and for every $x\in V_{\mathscr{G}}$ if $\mathscr{G}$ is a transitive graph in which each vertex has an odd number of nearest neighbors, e.g., the hexagonal lattice and homogeneous trees of odd degrees. See also ref. 1 for the model on the homogeneous tree of degree three. In this note, we consider the model on the triangular lattice and prove the following theorem.

Theorem 1. Let $\mathscr{G}$ be the triangular lattice and $J_{x,y} = +1$ for all $\{x,y\}\in E_{\mathscr{G}}$. Then for almost every σ^0 and τ and for every $x\in V_{\mathscr{G}}$, σ_x^t flips infinitely many times.

The idea of the proof of the theorem is from that of Theorem 2 in ref. 2. The key in the proof is the translation ergodicity along the three lines passing through each vertex. It is unknown if σ_x^t flips infinitely many times or only finitely many times when $\mathscr{G}$ is Z^d with $d\geqslant 3$, or a homogeneous tree of even degrees, or a hyperbolic graph (see ref. 3 for a definition) in which each vertex has an even number of nearest neighbors.

2. PROOF OF THE THEOREM

First of all, since the distributions P_τ and P_{σ^0} are translation-invariant, so is the product distribution $P_{\tau,\sigma^0} = P_\tau \times P_{\sigma^0}$. For a given vertex x, let A_x^+ (or A_x^-) be the event that $\sigma_x^\infty(\tau, \sigma^0)$ exists and equals $+1$ (or -1), and denote by I_x^+ (or I_x^-) the indicate function of this event. By translation-invariance and symmetry under the global spin flip $\sigma^0 \to -\sigma^0$, it follows that for all x, $P_{\tau,\sigma^0}(A_x^+) = P_{\tau,\sigma^0}(A_x^-) = p$ for some $p \in [0, 1/2]$. We wish to prove that $p = 0$.

Suppose $p > 0$. Fix a vertex a in the lattice. By translation-ergodicity, for each of the three lines passing through a there are, with P_{τ,σ^0}-probability one, infinitely many vertices x on *each* side of a such that A_x^- occurs. This implies that there exists vertices b, d and f on the three lines passing through a as shown in Fig. 1 such that

$$P_{\tau,\sigma^0}(A_a^+ A_b^- A_d^- A_f^-) > 0.$$

So there exists some t_0 such that, with strictly positive P_{τ,σ^0}-probability, $\sigma_a^t = +1$ and $\sigma_b^t = \sigma_d^t = \sigma_f^t = -1$ for all $t \geqslant t_0$. But this would at least require that the transition probabilities of the Markov process σ^t satisfies

$$\inf_{\sigma \in \Omega'} P_\tau(\sigma^{t+1} \notin \Omega' \mid \sigma^t = \sigma) = 0,$$

where Ω' is the set of spin configurations on the triangular lattice such that $\sigma_a = +1$ and $\sigma_b = \sigma_d = \sigma_f = -1$. Next, we will reach a contradiction by showing

$$\inf_{\sigma \in \Omega'} P_\tau(\sigma^{t+1} \notin \Omega' \mid \sigma^t = \sigma) > 0. \tag{2.1}$$

This will prove that $p = 0$.

Let R be the finite region enclosed in the polygon "*abcdef*." More precisely, R is the set of vertices inside (and on the boundary of) the polygon *abcdef*. Let $\Omega_R = \{-1, +1\}^R$ be the set of spin configurations in R and Ω'_R be the subset of Ω_R with $\sigma_a = +1$ and $\sigma_b = \sigma_d = \sigma_f = -1$. Because there are only finitely many elements in Ω'_R, in order to show (2.1) it is sufficient to show that for each $\sigma'_R \in \Omega'_R$

$$P_\tau(\sigma^{t+1} \notin \Omega' \mid \sigma^t|_R = \sigma'_R) > 0, \tag{2.2}$$

where $\sigma|_R$ is σ restricted on R.

For any $\sigma'_R \in \Omega'_R$, we call a Peierls contour (in the dual lattice) which separates plus and minus spins in R a domain wall. We claim that for any

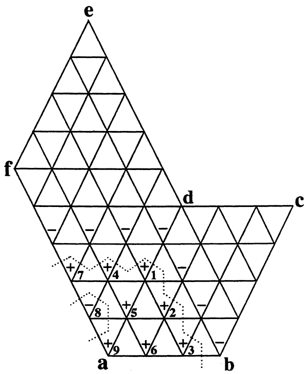

Fig. 1.

$\sigma'_R \in \Omega'_R$, there exists a domain wall which either (1) starts from line segment af and ends on line segment *ab*, or (2) starts from line segment *ab* and ends on line segment *bc*, or (3) starts from line segment *ad* and ends on line segment *cd*, or (4) starts from line segment *ad* and ends on line segment *de*, or else (5) starts from line segment af and ends on line segment ef. To prove the claim, first notice that the spins at vertices a and f are respectively $+1$ and -1. So along the line segment from a to f, there exists the first vertex at which the spin is -1. Call this vertex i_1 and the vertex right before it $i_1 - 1$. Then there is a domain wall starting at the midpoint between vertices $i_1 - 1$ and i_1. If this domain wall comes back to line segment af and ends at the midpoint between some $i_2 - 1$ and i_2, then the spin at i_2 must be $+1$. Call the first vertex after i_2 which has spin -1 as i_3. Then there is a domain wall starting at the midpoint between $i_3 - 1$ and i_3. If this domain wall ends on line segment af again, then repeat the procedure. Eventually, we can find the *first* (counting from vertex a to vertex f) domain wall which starts on line segment af and ends on either line segment *ab* or *bc* or *cd* or *de* or ef. If it ends on line segment *ab* or ef, then the proof is finished. If it ends on line segment *bc*, then it cuts region R into two parts: R_b, the part which contains vertex b, and $R - R_b$. Notice that the spins on the boundary of R_b along this domain wall are all $+1$, so any other domain wall in R_b can not intersect this domain wall. Find the *first* domain wall (counting from vertex b to vertex a) which starts from line segment *ba* and does not end on *ba*. Then this domain wall must end on *bc* because it can not end on af, since if it ends on af, then the domain wall which starts from af and ends on *bc* would not be the first one not ending on af (counting from vertex a to vertex f). Therefore, there exists a domain wall which starts on line segment *ab* and ends on line segment *bc*. If the domain wall starting on line segment af ends on line segment *cd*, then it crosses *ad* and hence there is a domain wall which starts on *ad* and ends on *cd*. Finally if the domain wall starting on *af* ends on *de*, then it cuts the parallelogram "*adef*" into two parts: R_d, the part which contains vertex *d*, and the rest of the diamond. Because of the way the domain wall is chosen, in R_d the vertex *d* is surrounded by +1 spins, ranging from vertex *a* to the beginning of the domain wall and then along the domain wall to the end of the domain wall on line segment *de*. So there must be a domain wall starting on line segment *ad* and ending on line segment *de*. This completes the proof of the claim.

We now turn to the proof of (2.2). Without loss of generality, suppose there exists a domain wall which starts on line segment *af* and ends on line segment *ab* (see Fig. 1). The other four cases listed in the claim can be argued similarly. This domain wall cuts R into two parts: R_a, the part which contains vertex *a*, and $R - R_a$. We first give an order to the vertices in R_a according to the following rule: order the vertex which is farthest from line *af* first; if two or more vertices have the same distance from line *af*, then order the one which is farthest from line *ab* first. Then with positive P_τ-probability there is some sequence of clock rings (according to the order prescribed above) and coin toss outcomes within R that will move the domain wall towards vertex *a* so that $\sigma_a^{t+1} = -1$. For example, in Fig. 1, the clocks rings in the order of 1, 2, 3, 4, 5, 6, 9. After the clock at 4 rings, the domain wall encloses only vertices 5, 6, and 9. From here it is further moved towards vertex *a* and finally the spin at *a* is changed from +1 to −1. This completes the proof of the theorem. ▮

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